Self-catalytic conversion of pure quantum states

Cristhiano Duarte,[1, 2, *] Raphael C. Drumond,[2, †] and Marcelo Terra Cunha[2, 3, ‡]

[1] *Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Juiz de Fora, CEP 36036-330, Juiz de Fora, Minas Gerais, Brazil.*

[2] *Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, CP 702, CEP 30123-970, Belo Horizonte, Minas Gerais, Brazil.*

[3] *Departamento de Matemática Aplicada, Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Cidade Universitária Zeferino Vaz, CEP 13083-970, Campinas, São Paulo, Brazil.*

(Dated: December 14, 2015)

Conversion of entangled states under (Stochastic) Local Operations and Classical Communication admits the phenomenon of catalysis. Here we explore the possibility of a copy of the initial state itself to perform as a catalyst, which we call a self-catalytic process. We show explicit examples of self-catalysis. Necessary and sufficient conditions for the phenomenon to take place are discussed. We numerically estimate how frequent it is and we show that increasing the number of copies used as catalyst can increases the probability of conversion, but do not make the process deterministic. By the end we conjecture that under LOCC the probability of finding a self-catalytic reaction *do not* increase monotonically with the dimensions whereas SLOCC, *does* increase.

Keywords: Catalysis, Convertibility, LOCC, SLOCC, Self-Catalisys.

I. INTRODUCTION

The conversion between bipartite or multipartite quantum states through local operations is a central concept of entanglement theory. For instance, it is the criterion used to classify entanglement, namely, two states have equivalent entanglement if they can be converted to each other [1]. Moreover, it is usual and natural to consider a state more entangled than other when the first can *access* the former by the allowed transformations [2]. Two important sets of allowed transformations are the deterministic *local operations and classical communication* (LOCC) and its stochastic version (SLOCC), where the transformation only needs to succeed with positive probability [3]. Although this hierarchisation is relatively simple for bipartite pure states [2, 4], it is quite involved if mixed or multipartite states are considered [5–7].

The problem of convertibility of bipartite pure states has been solved by Nielsen[2], using the concept of majorisation [8]. However, Jonathan and Plenio discovered a surprising effect [9]. They have shown the existence of pairs of states which are not directly inter-convertible but such that their conversion is possible if another (necessarily entangled) state is attached to them. That is, the pair of states have incomparable entanglement but they may become ordered if an extra system is attached to the original ones. Such extra state that makes a transformation possible, without being consumed, is called a *catalyst*. More recently, the problem of convertibility has received experimental attention [10] and has also been connected to basic results in thermodynamics [11] and phase transitions [12].

In this paper we explore the following question: can one of the states of a non inter-convertible pair be used as a catalyst? We answer this question positively, providing explicit examples. We also explore how common such processes are for low dimensional systems. An interesting situation is when a state $|\Psi\rangle$ is not able of self-catalysing a transformation, but a number of copies, $|\Psi\rangle^{\otimes n}$, is. We exhibit examples where more than one copy is required and study how augmenting the number of copies can increase the probability of conversion.

In section II we review basic notions of state conversion and catalysis under LOCC. In section III we show explicit examples of self-catalytic processes, explore, through numerical analysis, how frequent they are under random samples of incomparable pairs and how it depends on the size of the systems. Section IV review the notions of probabilistic catalysis under SLOCC, while the natural questions of self-catalysis under SLOCC are discussed in section V. We close the main text with final remarks and further problems in section VI.

*Electronic address: cristhiano@mat.ufmg.br
†Electronic address: raphael@mat.ufmg.br
‡Electronic address: tcunha@ime.unicamp.br

II. CATALISYS

We say that a bipartite state $|\alpha\rangle \in H_A \otimes H_B$ *access* a state $|\beta\rangle \in H'_A \otimes H'_B$, if there is some LOCC operation, represented by a completely positive trace preserving (CPTP) map Λ, such that $\Lambda(|\alpha\rangle\langle\alpha|) = |\beta\rangle\langle\beta|$, where H_A, H'_A, H_B and H'_B are finite dimensional Hilbert spaces and Λ maps density operators acting on $H_A \otimes H_B$ to density operators acting on $H'_A \otimes H'_B$. In such case, we write $|\alpha\rangle \to |\beta\rangle$. If there is no LOCC operation able to convert $|\alpha\rangle$ to $|\beta\rangle$, we shall write $|\alpha\rangle \nrightarrow |\beta\rangle$.

For example, for a pair of qubits, the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ can access any two-qubit pure state. Indeed, naming our qubits A and B, and writing the target state as $|\beta\rangle = a|00\rangle + b|11\rangle$, we can, for example, make A unitarily interact with an auxiliary qubit A', so that $|0_A 0_{A'}\rangle \mapsto a|0_A 0_{A'}\rangle + b|1_A 1_{A'}\rangle$ and $|1_A 0_{A'}\rangle \mapsto a|0_A 1_{A'}\rangle + b|1_A 0_{A'}\rangle$. Then, $[(|0_A 0_B\rangle + |1_A 1_B\rangle)/\sqrt{2}]\,|0_{A'}\rangle \mapsto (1/\sqrt{2})(a|0_A 0_B\rangle + b|1_A 1_B\rangle)|0_{A'}\rangle + (1/\sqrt{2})(a|0_A 1_B\rangle + b|1_A 0_B\rangle)|1_{A'}\rangle$. The lab with qubit A can make a measurement on the computational basis of the auxiliary A' and send the result to the lab holding qubit B. If the result is 0, they already share the desired state, while the result being 1, a NOT operation, $|0_B\rangle \mapsto |1_B\rangle, |1_B\rangle \mapsto |0_B\rangle$, can be applied to qubit B to also leave the system AB in the desired state. This result generalises for a pair of qu*d*its in the state $|\Phi_d^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$, which justifies $|\Phi_d^+\rangle$ to be called a maximally entangled state.

Nielsen, in the seminal paper [2], provided a simple necessary and sufficient condition for determining whether a general bipartite state $|\alpha\rangle$ can access a state $|\beta\rangle$ in terms of their corresponding Schmidt vectors [1]:

Theorem 1 (Nielsen Criterion)**.** *If $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$ and $\vec{\beta} = (\beta_1, ..., \beta_m)$ are the (non-increasing) ordered Schmidt vectors of $|\alpha\rangle$ and $|\beta\rangle$, respectively, we have $|\alpha\rangle \to |\beta\rangle$ if, and only if,*

$$\sum_{l=1}^{k} \alpha_l \le \sum_{l=1}^{k} \beta_l \tag{1}$$

for $1 \le k \le \min\{n, m\}$.

By an ordered Schmidt vector $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, we mean that $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$. In this work, we shall always assume that Schmidt vectors are ordered. When vectors $\vec{\alpha}$ and $\vec{\beta}$ satisfy conditions (1) we say that $\vec{\alpha}$ is majorised by $\vec{\beta}$ and write $\vec{\alpha} \preceq \vec{\beta}$.

Revisiting the example above, we have $(1/2, 1/2)$ for the ordered Schmidt vector of $|\Phi^+\rangle$ and $(|a|^2, |b|^2)$ for $a|00\rangle + b|11\rangle$, assuming $|a| \ge |b|$, it is straightforward to apply the criterion and verify that $|\Phi^+\rangle \to [a|00\rangle + b|11\rangle]$.

A consequence of the criterion is the existence of pair of states $|\alpha\rangle$ and $|\beta\rangle$ such that $|\alpha\rangle \nrightarrow |\beta\rangle$ and $|\beta\rangle \nrightarrow |\alpha\rangle$. Actually, the only case where such an order is total is for two qubits. For instance, consider a state of two four-level systems with Schmidt vector $(0.4, 0.4, 0.1, 0.1)$ and a two-qutrit state with Schmidt vector $(0.5, 0.25, 0.25)$. Indeed,

$$0.4 = \alpha_1 < \beta_1 = 0.5, \tag{2a}$$

$$0.4 + 0.4 = \alpha_1 + \alpha_2 > \beta_1 + \beta_2 = 0.5 + 0.25. \tag{2b}$$

In Ref. [9], the authors surprisingly show that it is possible to circumvent such non-accessibility between these states by making the parts to share an entangled state $|\kappa\rangle$ such that $|\alpha\rangle \otimes |\kappa\rangle \to |\beta\rangle \otimes |\kappa\rangle$ (see Figure 1). Since the state $|\kappa\rangle$ allows for a previously forbidden conversion, but at the end of the process it remains unaltered, it is called a *catalyst*. In this sense, we say that $|\alpha\rangle$ *$\vec{\kappa}$-access* $|\beta\rangle$ when $|\alpha\rangle \nleftrightarrow |\beta\rangle$, but $|\alpha\rangle \otimes |\kappa\rangle \to |\beta\rangle \otimes |\kappa\rangle$. In this specific example, the two-qubit state $|\kappa\rangle$ with Schmidt vector $\vec{\kappa} = (0.6, 0.4)$ is a catalyst.

Nielsen's criterion ensures that all information about the (possibility of) conversion is contained in the Schmidt vectors. Therefore, we can explore our knowledge regarding probability vectors to provide more examples of catalysts. If $\vec{\alpha} = (0.5, 0.4, 0.05, 0.05)$, $\vec{\beta} = (0.7, 0.15, 0.15)$, $\vec{\kappa_1} = (0.7, 0.3)$, and $\vec{\kappa_2} = (0.75, 0.25)$, we obtain:

$$\vec{\alpha} \nleftrightarrow \vec{\beta} \tag{3a}$$

$$\vec{\alpha} \otimes \vec{\kappa_1} \to \vec{\beta} \otimes \vec{\kappa_1} \tag{3b}$$

$$\vec{\alpha} \otimes \vec{\kappa_2} \to \vec{\beta} \otimes \vec{\kappa_2}. \tag{3c}$$

This is a good example of non-unicity of catalysts. Indeed, for a given forbidden transition, $|\alpha\rangle \nrightarrow |\beta\rangle$, and fixed local dimensions for the catalysts, the set of allowed vectors $\vec{\kappa}$ is a polytope [13].

Another interesting geometric fact is that catalysis is possible for every bipartite scenario, starting from 4×3, *i.e.*, for all effective dimensions $m \ge n$, $m \ge 4$, and $n \ge 3$ it is possible to choose $\vec{\alpha} = (\alpha_1, ..., \alpha_m) \nleftrightarrow (\beta_1, ..., \beta_n) = \vec{\beta}$

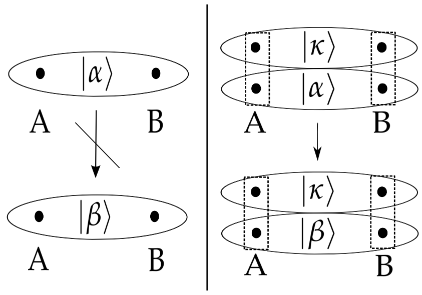

FIG. 1: Although the conversion $|\alpha\rangle \to |\beta\rangle$ is not allowed under LOCC, the state $|\kappa\rangle$ can be used (but not consumed) to make it viable.

$\vec{\alpha}$	$\vec{\beta}$	$\sharp$Copies (N)
$(0.900, 0.081, 0.010, 0.009)$	(0.950,0.030,0.020)	$N = 1$
$(0.900, 0.088, 0.006, 0.006)$		$N = 2$
$(0.908, 0.080, 0.006, 0.006)$		$N = 3$
$(0.918, 0.070, 0.006, 0.006)$	$(0.950, 0.030, 0.020)$	$N = 4$
$(0.925, 0.063, 0.006, 0.006)$		$N = 5$
$(0.928, 0.060, 0.006, 0.006)$		$N = 6$
$(0.900, 0.081, 0.010, 0.009)$	(0.950,0.030,0.019,0.001)	$N = 1$

TABLE I: Number N of copies required to make $|\alpha\rangle^{\otimes N}$ a deterministic catalyst for the process $|\alpha\rangle \to |\beta\rangle$.

with catalyst $\vec{\kappa}$. The essential step, after the previously described examples, is that given $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\kappa}$, we generically can increase dimensions by one and construct a forbidden transition $\vec{\alpha'} \nrightarrow \vec{\beta'}$ with the same catalyst $\vec{\kappa}$, by using $\vec{\alpha'} = (\alpha_1, ..., \alpha_n - \epsilon, \epsilon)$ and $\vec{\beta'} = (\beta_1, ..., \beta_n - \epsilon', \epsilon')$ with small enough $\epsilon > 0$ and $\epsilon' \geq 0$. This allow us to construct examples for all such m and n.

III. SELF-CATALYSIS

In this section we address the main question of this paper for the case of LOCC convertibility: can a bipartite quantum state be itself the catalyst of a forbidden conversion? To be more precise, is there a forbidden conversion $|\alpha\rangle \nrightarrow |\beta\rangle$, such that $|\alpha\rangle\otimes|\alpha\rangle \to |\beta\rangle\otimes|\alpha\rangle$? Refining a little bit more, is it possible that we still have $|\alpha\rangle\otimes|\alpha\rangle \nrightarrow |\beta\rangle\otimes|\alpha\rangle$, but a larger number of copies of $|\alpha\rangle$ would do the job, *i.e.*, $|\alpha\rangle \otimes |\alpha\rangle^{\otimes N} \to |\beta\rangle \otimes |\alpha\rangle^{\otimes N}$ for some $N > 1$?

In Table I we list examples to answer affirmatively these questions. The first is an example of self-catalysis from a two-ququart state to a two-qutrit state. Then, for the same scenario, there is a list of multi-copy self-catalysis, with the corresponding minimal number of copies. At the last row, we come back to single-copy self-catalysis from a two-ququart state to another two-ququart state.

Remark. When $|\alpha\rangle$ has 4 non-zero Schmidt coefficients and $|\beta\rangle$ has 3 non-zero Schmidt coefficients, the phenomenon of self-catalysis can be noted, that is, even in the minimal dimensions to occur catalysis (see [9]) the self-catalysis can also carry out. Geometrically, this mean that even is the smallest dimension (most restrictive) case, there are cases where the source state $\vec{\alpha}$ belongs to the polytope of catalysts for the reaction $\vec{\alpha} \nrightarrow \vec{\beta}$. Moreover, it is possible to construct examples for any scenario through a similar argument presented at the Section II.

A. Stability under small perturbations

It is important to mention that the phenomenon of self-catalysis, as it happens with catalysis, is generically robust against small perturbations of the state vectors involved. That is, suppose that one is aiming to perform a self-catalytic process $|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle$, but it actually implements states $|\alpha'\rangle$, $|\beta'\rangle$, where $|\alpha'\rangle \approx |\alpha\rangle$ and $|\beta'\rangle \approx |\beta\rangle$. Our claim is: *generically, if* $|\alpha\rangle$ *α-access* $|\beta\rangle$*, then* $|\alpha'\rangle$ *α'-access* β'. It is easy to see that this will be true, depending only on the inequalities implying that $|\alpha\rangle \nrightarrow |\beta\rangle$, as well as those assuring that $|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle$, all be strict (except the last one, which is granted by normalisation). Denote by $|\vec{\lambda}|$ some norm (*e.g.*, Euclidean) of the vector $\vec{\lambda}$ and n, m

the sizes of Schmidt vectors $\vec{\alpha}, \vec{\beta}$. Now, assuming that we have, for some l, $\sum_{j=1}^{k} \alpha_j < \sum_{j=1}^{k} \beta_j$ for $1 \leq k \leq l$ and $\sum_{j=1}^{k} \alpha_j > \sum_{j=1}^{k} \beta_j$ for $l < k \leq n-1$, the same set of inequalities will hold for the entries of vectors $\vec{\alpha'}$ and $\vec{\beta'}$ if $|\vec{\alpha} - \vec{\alpha}'|, |\vec{\beta} - \vec{\beta}'| < \epsilon$, as long as ϵ is small enough. This implies that $|\alpha'\rangle$ and $|\beta'\rangle$ are still incomparable. A similar reasoning can be applied when the incomparability of vectors $\vec{\alpha}$ and $\vec{\beta}$ is due to two or more changes of signs in the inequalities. In the same way, $\sum_{j=1}^{k} (\alpha \otimes \alpha)_j^{\downarrow} < \sum_{j=1}^{k} (\beta \otimes \alpha)_j^{\downarrow}$ for $k < nm-1$, imply $\sum_{j=1}^{k} (\alpha' \otimes \alpha')_j^{\downarrow} < \sum_{j=1}^{k} (\beta' \otimes \alpha')_j^{\downarrow}$, if $|\vec{\alpha} - \vec{\alpha}'|, |\vec{\beta} - \vec{\beta}'| < \epsilon$, for small enough ϵ, which proves the claim. Naturally, the argument includes the well-motivated situation when $|\beta'\rangle = |\beta\rangle$ as a special case.

B. Self-Catalysis under LOCC for random Schmidt vectors

We have numerically investigated how usual the phenomenon of self-catalysis among pairs of incomparable bipartite states is. Fixing the sizes of $\vec{\alpha}$ and $\vec{\beta}$, we randomly sample pairs of such vectors until we find incomparable ones. The sampling of each vector is done by uniformly sorting unitary vectors in $\mathbb{C}^n \otimes \mathbb{C}^n$, *i.e.* sorting according to the *Haar Measure* in the respective state spaces [21, 22], and then calculating the correspondent Schmidt vector. After finding a pair of incomparable states, we test whether the first of the vectors can be used as a catalyst for the conversion. The results show that for this method of sampling and for small dimensional systems, the phenomenon is actually *atypical*. Moreover, the numerical estimations seem to imply that the phenomenon of self-catalysis is atypical in any dimension.

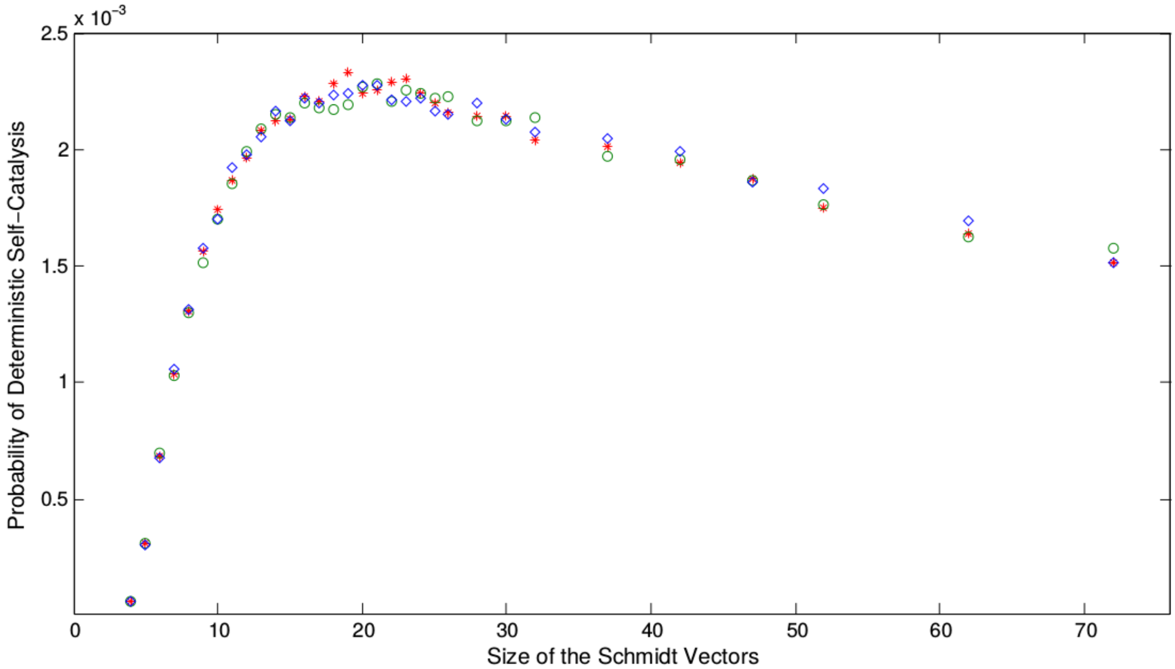

FIG. 2: Probability of finding a pair of states exhibiting self-catalysis as function of the dimension of each system, that is $\mathbb{P}(|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle \;|\; |\alpha\rangle \not\leftrightarrow |\beta\rangle)$. Each symbol means an average over a distinct set of random choices. For each size explored there are three different symbols, which indicates a reasonable stability in this sampling process.

Figure 2 shows a numerical estimation for $\mathbb{P}(|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle \;|\; |\alpha\rangle \not\leftrightarrow |\beta\rangle)$, that is, the conditional probability of self-catalysis given a pair of incomparable Schmidt vectors, as a function of their sizes. Note that this probability increases with the size of the Schmidt vectors, until sizes about 20 and then starts to slowly decrease, but in fact for any dimension its order of magnitude shows that the phenomenon is present, but rare.

We believe the global maximum present in the Figure 2 can be explained by the algebraic character of the comparison between vectors and the concentration of measure phenomenon. In order to $|\alpha\rangle \otimes |\alpha\rangle$ access $|\alpha\rangle \otimes |\beta\rangle$, all the corresponding inequalities of Eq. (1) must be satisfied. Then, in one hand, by increasing the dimension of the vectors, this comparison become harder to be valid (more inequalities have to be obeyed). On the other hand, it is known that a measure concentration phenomenon takes place for increasing dimensionality, in the sense that the Schmidt

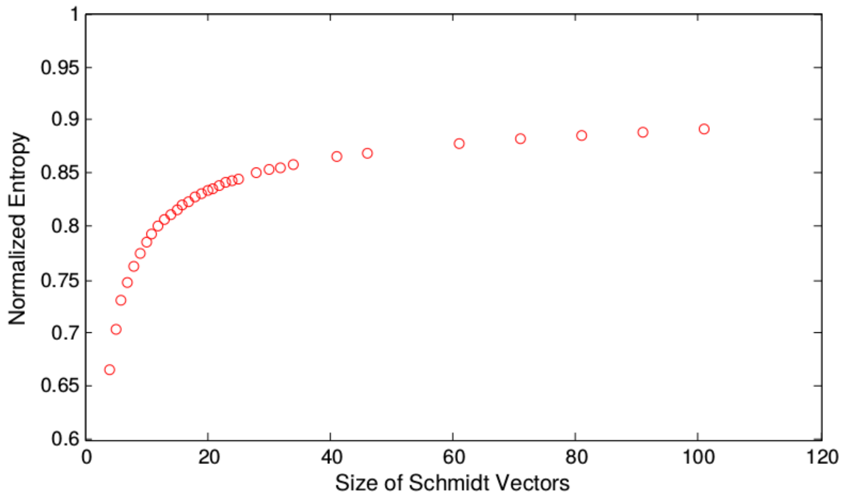

FIG. 3: Mean normalized entropy for Schmidt vectors sorted by Haar measure. Here we have explored the behavior of the entropy until effective dimension 100.

vectors become typically closer and closer to the constant vector $(\frac{1}{d}, ..., \frac{1}{d})$. This phenomenon can be highlighted by the average normalized entropy of the sorted vectors: if this average is close to 1, it means that most vectors are close to the constant vector. This average value is know to be exactly $\frac{1}{\ln d}[\sum_{k=d+1}^{d^2} \frac{1}{k} - \frac{d-1}{2d}]$, as conjectured by Page [19] and latter proved by Foong and Kanno [20]

If the vectors of the pair are incomparable, that is, some inequalities of Eq. (1) are not satisfied, but both are close to the constant vector, it will be easier for the catalyst to make the transition possible, since the corresponding inequalities for the vectors with the catalyst attached will be easier to be satisfied.

We can note that the concentration of measure increases very fast at low dimensions (see Figure 3), helping the possibility of "organising" the Schmidt vectors with the catalyst attached, which justifies the corresponding increasing in the probability for self-catalysis. However, around dimension 20 the concentration happens much slower and presumably is not fast enough to overcome the size effect (which makes majorisation more difficult), so the probability of finding a self-catalytic pair decreases after this point.

Summing up, the probability seems to be a smooth function of the dimension, reaching its higher value for dimensions of each Schmidt vector close to 20 and, apparently, converging to a value considerably smaller than 1.

Remark 1. *We have restricted the analysis above to events where $|\alpha\rangle \not\leftrightarrow |\beta\rangle$. For large dimensions we have checked that this event* is *typical. Indeed, in such regime the entries of vectors $\vec{\alpha}$ and $\vec{\beta}$, before ordering, are essentially concentrated random variables fluctuating around a fixed value. If we look, then, to Eq.* (1) *we see that there is a good chance for the sign of the inequality to change as we vary the index k, implying that the states are incomparable. Therefore, we expect that $\mathbb{P}(|\alpha\rangle \not\leftrightarrow |\beta\rangle) \approx 1$, for $n \gg 1$.*

IV. CATALYSIS UNDER SLOCC

It is possible to generalize the concept of accessibility if we allow for non-deterministic processes. In this case, we can look for the probability $P_{SLOCC}(|\alpha\rangle \to |\beta\rangle)$, or $P_S(|\alpha\rangle \to |\beta\rangle)$ for short, of having the state conversion $|\alpha\rangle \to |\beta\rangle$ under the best local strategy, *i.e.*, optimizing P under the conditions defining SLOCC. It is interesting to recall that the famous result on inconvertibility between W and GHZ states refers to such conditions on the multipartite scenario [14].

As shown by Vidal [15], for the bipartite case, the Schmidt vectors also encode this maximal probability of conversion, $P_S(|\alpha\rangle \to |\beta\rangle)$ through

Theorem 2. *Let $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$ and $\vec{\beta} = (\beta_1, ..., \beta_n)$ be ordered Schmidt vectors for states $|\alpha\rangle$ and $|\beta\rangle$, assuming*

$\alpha_n, \beta_n > 0$. *Define* $E_k(\lambda) = 1 - \sum_{l=1}^{k-1} \lambda_l$. *Then, the optimal transformation probability is given by*

$$P_S(|\alpha\rangle \to |\beta\rangle) = \min_{1\leq k\leq n} \left\{ \frac{E_k(\alpha)}{E_k(\beta)} \right\}. \tag{4}$$

For instance, if

$$\vec{\alpha} = (0.6, 0.2, 0.2) \text{ and } \vec{\beta} = (0.5, 0.4, 0.1) \tag{5}$$

are the Schmidt vectors for the two-qutrit states $|\alpha\rangle$ and $|\beta\rangle$, respectively, we get:

$$P_S(|\alpha\rangle \to |\beta\rangle) = 0.8, \tag{6a}$$
$$P_S(|\beta\rangle \to |\alpha\rangle) = 0.5. \tag{6b}$$

The following Proposition shows that the optimal probability of conversion P_S attains 1 precisely when $|\alpha\rangle$ access $|\beta\rangle$.

Proposition 1. *Let* $\vec{\alpha}$ *and* $\vec{\beta}$ *be a pair of random independent Schmidt vectors with same size* n*, then the event* $\{|\alpha\rangle \to |\beta\rangle\}$ *is equal to event* $\{P_S(|\alpha\rangle \to |\beta\rangle) = 1\}$. *In particular* $\mathbb{P}(|\alpha\rangle \to |\beta\rangle) = \mathbb{P}(P_S(|\alpha\rangle \to |\beta\rangle) = 1)$.

Proof. Suppose that $|\alpha\rangle \to |\beta\rangle$, thus:

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i, \ \forall k \ \in \{1, 2, ..., n\}. \tag{7}$$

Then $E_k(\alpha) = 1 - \sum_{i=1}^{k-1} \alpha_k \geq 1 - \sum_{i=1}^{k-1} \beta_k = E_k(\beta)$, $\forall\ k \in \{1, 2, ..., n-1\}$ with the equality holding if $k = 1$, therefore $P_S(|\alpha\rangle \to |\beta\rangle) = 1$. The converse is similar. $\square$

Note that from our considerations at the end of section III B we expect that $\mathbb{P}(|\alpha\rangle \to |\beta\rangle) \approx 0$ (using LOCC) for large n, since the event $\{|\alpha\rangle \to |\beta\rangle\}$ is in the complement of $\{|\alpha\rangle \not\leftrightarrow |\beta\rangle\}$.

We numerically estimate the SLOCC average rate of conversion between random states. For this, we generate incomparable Schmidt vectors following the *Haar measure* and calculate the probabilities of conversion from the first to the second and also the maximum conversion probability. The results shown in Figure 4 clearly shows that, given a random pair α, β, it is very common to have a large probability of conversion from some of them to the other, that is $\mathbb{E}[\max\{P_S(|\alpha\rangle \to |\beta\rangle\,, P_S(|\beta\rangle \to |\alpha\rangle)\}] \gtrsim 0.8$. Moreover we have a smaller, but still significant, average probability of conversion $\mathbb{E}[P_S(|\alpha\rangle \to |\beta\rangle)]$, slightly below 0.6.

Also in the probabilistic scenario the presence of an extra state can improve the probability of conversion between two states [16]. As a chemical catalyst, this extra state is used, but not consumed, to increase the rate (probability) of a reaction (conversion). In the above example, if $\vec{\kappa} = (0.65, 0.35)$, we arrive at:

$$\vec{\alpha} \otimes \vec{\kappa} = (0.39, 0.21, 0.13, 0.13, 0.07, 0.07), \tag{8a}$$
$$\vec{\beta} \otimes \vec{\kappa} = (0.325, 0.26, 0.175, 0.14, 0.065, 0.035). \tag{8b}$$

Therefore $P_S(|\alpha\rangle \otimes |\kappa\rangle \to |\beta\rangle \otimes |\kappa\rangle) \simeq 0.904$, and $|\kappa\rangle$ can be viewed as a *probabilistic-catalyst* in the stochastic scenario for the conversion that starts in α and ends in β, despite the fact that $P_S(|\beta\rangle \otimes |\kappa\rangle \to |\alpha\rangle \otimes |\kappa\rangle) = 0.5$, and then $|\kappa\rangle$ does not increases the probability of conversion for the transformation that starts in β and ends in α. In this sense Jonathan and Plenio pointed out [9] that if $P_S(|\alpha\rangle \to |\beta\rangle)$, under the best local strategy, is equal to α_n/β_n, then this probability can not be increased by the presence of any catalyst state. Feng *et. al.* [16] improved this result, obtaining the following theorem:

Theorem 3. *Let* $\vec{\alpha}$ *and* $\vec{\beta}$ *be two* $n-$*dimensional probability vectors written in non-increasing order. There is a probability vector* $\vec{\kappa}$ *such that* $P_S(\vec{\alpha} \otimes \vec{\kappa} \to \vec{\beta} \otimes \vec{\kappa}) > P_S(\vec{\alpha} \to \vec{\beta})$ *if, and only if,*

$$P_S(\vec{\alpha} \to \vec{\beta}) < \min\left\{ \frac{\alpha_n}{\beta_n}, 1 \right\}. \tag{9}$$

A deeper connection between LOCC catalysis and its stochastic counterpart connects the probability of occurrence of the event $\{\alpha \to \beta\}$ and the maximal probability of stochastic conversion, $P_S(|\alpha\rangle \otimes |\phi\rangle \to |\beta\rangle \otimes |\phi\rangle)$.

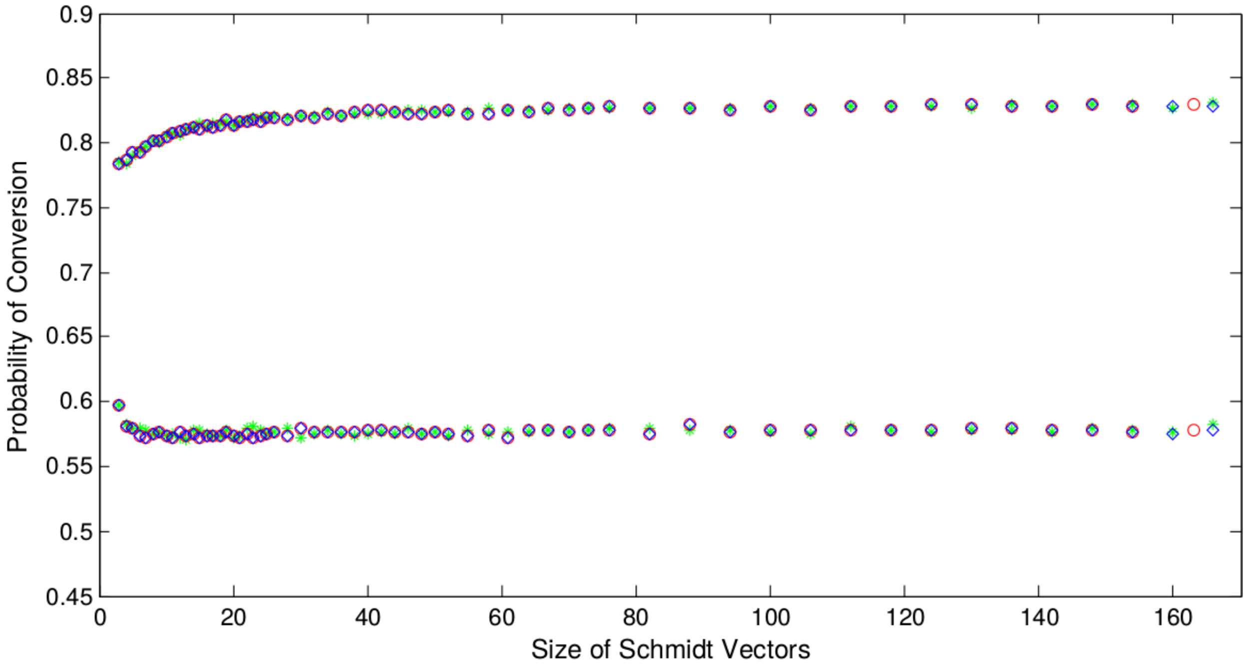

FIG. 4: Sampled probability of conversion under SLOCC for randomly chosen (following the Haar measure) incomparable states as function of the dimension of each system. Each symbol represents an average over a distinct set of randomly chosen pairs. There are two sets of three symbols for each explored dimension. For each dimension, the smaller results consider conversion from the first to the second $\mathbb{E}[P_S(|\alpha\rangle \to |\beta\rangle)]$, and the larger the maximum conversion rate $\mathbb{E}[\max\{P_S(|\alpha\rangle \to |\beta\rangle, P_S(|\beta\rangle \to |\alpha\rangle)\}]$.

Proposition 2. *Let $\vec{\alpha}$ and $\vec{\beta}$ be a pair of random independent Schmidt vectors with same size n. Then*

$$\mathbb{P}[\sup_{\phi}\{P_S(|\alpha\rangle \otimes |\phi\rangle \to |\beta\rangle \otimes |\phi\rangle)\} > P_S(|\alpha\rangle \to |\beta\rangle)] \quad \geq \quad \frac{1}{2} - \mathbb{P}(|\alpha\rangle \to |\beta\rangle). \tag{10}$$

Proof.

$$\begin{aligned} \mathbb{P}[P_S(|\alpha\rangle \otimes |\phi\rangle \to |\beta\rangle \otimes |\phi\rangle) > P_S(|\alpha\rangle \to |\beta\rangle)] &= 1 - \mathbb{P}\{P_S(|\alpha\rangle \to |\beta\rangle) = \min(\alpha_n/\beta_n, 1)\} && \text{(11a)}\\ &\geq 1 - \mathbb{P}\{P_S(|\alpha\rangle \to |\beta\rangle) = \alpha_n/\beta_n\} - \mathbb{P}\{|\alpha\rangle \to |\beta\rangle\} && \text{(11b)}\\ &\geq \frac{1}{2} - \mathbb{P}\{|\alpha\rangle \to |\beta\rangle\}. && \text{(11c)} \end{aligned}$$

Where (11a) comes from Thm 3, (11b) from set theory and Proposition 1, and finally (11c) from $\{P_S(|\alpha\rangle \to |\beta\rangle) = \alpha_n/\beta_n\} \subseteq \{\alpha_n \leq \beta_n\}$. □

Remark 2. *From Remark 1 we know that as n grows, $\mathbb{P}(|\alpha\rangle \to |\beta\rangle) \approx 0$, so we conclude from Proposition 2 that* $\mathbb{P}[P_S(\vec{\alpha} \to \vec{\beta}) < \min\left\{\frac{\alpha_n}{\beta_n}, 1\right\}] \gtrsim 1/2$.

In Ref. [17] a necessary and sufficient condition for a state $|\kappa\rangle$ to work as a probabilistic catalyst was provided:

Theorem 4. *Suppose that $\vec{\alpha}$ and $\vec{\beta}$ are two non-increasingly ordered n–dimensional probability vectors, and $P(\vec{\alpha} \to \vec{\beta}) < \min\left\{\frac{\alpha_n}{\beta_n}, 1\right\}$. Define*

$$L = \left\{ l; 1 < l < n, \text{ and } P(\vec{\alpha} \to \vec{\beta}) = \frac{E_l(\vec{\alpha})}{E_l(\vec{\beta})} \right\}. \tag{12}$$

Then a non-increasingly ordered k-dimensional probability vector $\vec{\kappa}$ serves as a probabilistic catalyst for the conversion from $|\alpha\rangle$ to $|\beta\rangle$ if, and only if, for all $r_1, r_2, ..., r_k \in L \cup \{n+1\}$ satisfying $r_1 \geq r_2 \geq ... \geq r_k \neq n+1$, there exist i and

j, with $1 \leq j < i \leq k$, such that

$$\frac{\kappa_i}{\kappa_j} < \frac{\beta_{r_j}}{\beta_{r_i-1}} \text{ or } \frac{\kappa_i}{\kappa_j} > \frac{\beta_{r_j-1}}{\beta_{r_i}}. \tag{13}$$

By definition, whenever one of the inequalities (13) includes an index $n+1$, it is considered to be violated, so the other one must necessarily be satisfied.

We should stress a couple of facts about the set L: first of all, note that it is a key ingredient for identifying catalysts for a given conversion, since in a certain sense it determines which indexes are really important for the comparison between $\vec{\beta}$ and $\vec{\kappa}$. Secondly observe that typically L has only one element l, *i.e.*, the minimum which determines the probability of conversion (see Thm. 2) is non-degenerate.

V. SELF-CATALYSIS UNDER SLOCC

The phenomenon of self-catalysis can also take place when considering conversions under SLOCC. Namely, if a conversion $|\alpha\rangle \to |\beta\rangle$ takes place with optimal probability $0 < p < 1$, it can be the case that the optimal probability for $|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle$ be $p' > p$. Indeed, for the same Schmidt vectors $\vec{\alpha}$ and $\vec{\beta}$ given by Eq. (5), there is a gain in the probability of conversion if we use the state $|\alpha\rangle$ itself as a catalyst. Using Eq. (4), we have:

$$P_S(|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle) \simeq 0.889 > 0.800 = P_S(|\alpha\rangle \to |\beta\rangle). \tag{14}$$

As it happens in the context of LOCC operations, the conversion between states can depend on the number of attached copies of $|\alpha\rangle$. Table II shows, for the example we are considering (given by Eq. (5)), how the probability of conversion increases with the number of copies of $|\alpha\rangle$ to be used as catalyst.

This example may suggest that, by increasing the number of copies of $|\alpha\rangle$, the probability of conversion approaches one. This is not always the case, however. Note that we can bound from above the probability in Thm. 2 for the pair $|\alpha\rangle \otimes |\alpha\rangle^{\otimes N}$ and $|\beta\rangle \otimes |\alpha\rangle^{\otimes N}$, since $P_S(|\alpha\rangle \otimes |\alpha\rangle^{\otimes N} \to |\beta\rangle \otimes |\alpha\rangle^{\otimes N}) \leq \alpha_n/\beta_n$, for all $N \geq 1$. Therefore, as long as $\alpha_n/\beta_n < 1$, no matter how many copies of $|\alpha\rangle$ we have, the probability of conversion will not exceed α_n/β_n. The previous reasoning allows us to state the following Proposition:

Proposition 3. *Let $\vec{\alpha}$ and $\vec{\beta}$ be a pair of ordered Schmidt vectors with n non-null components. If $\alpha_n/\beta_n < 1$, then*

$$P_S(|\alpha\rangle \otimes |\alpha\rangle^{\otimes N} \to |\beta\rangle \otimes |\alpha\rangle^{\otimes N}) \leq \frac{\alpha_n}{\beta_n}, \ \forall\ N \geq 0. \tag{15}$$

For example, given $\vec{\alpha} = (0.60, 0.21, 0.10, 0.09)$ and $\vec{\beta} = (0.55, 0.25, 0.10, 0.10)$, we have a pair of incomparable states with $P_S(|\alpha\rangle \to |\beta\rangle) = 0.88$, but since $\alpha_4/\beta_4 = 0.9$, the probability of conversion under SLOCC using self-catalysis is limited by 0.9 and indeed, for this case, $N = 1$ is already optimal, since $P_S(|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle) = 0.9$. Analogously, for the pair $\vec{\alpha'} = (0.40, 0.34, 0.15, 0.11)$ and $\vec{\beta'} = (0.50, 0.21, 0.17, 0.12)$, Table III shows the behavior of $P_S(|\alpha'\rangle \otimes |\alpha'\rangle^{\otimes N} \to |\beta'\rangle \otimes |\alpha'\rangle^{\otimes N})$ with the number N of copies of $|\alpha'\rangle$, and since $\alpha'_n/\beta'_n = 0.91667 < 1$ the probability of conversion may increase, but can not reach 1. Moreover, there is the case (see Table IV) where, for a given pair $\vec{\alpha}, \vec{\beta}$ of incomparable Schmidt vectors, the probability of conversion increases monotonically with respect to N and, for an $N_0 < \infty$, $P_S(|\alpha\rangle \otimes |\alpha\rangle^{\otimes N_0} \to |\beta\rangle \otimes |\alpha\rangle^{\otimes N_0}) = 1$ and, by Proposition 1, $|\alpha\rangle \otimes |\alpha\rangle^{\otimes N_0} \to |\beta\rangle \otimes |\alpha\rangle^{\otimes N_0}$.

A particular case of Theorem 4 allows us to obtain a necessary and sufficient condition for having probabilistic self-catalysis for a single copy:

Criterion *Let $\vec{\alpha}$ and $\vec{\beta}$ be two n–dimensional Schmidt vectors with $P_S(\vec{\alpha} \to \vec{\beta}) < \min\left\{\frac{\alpha_n}{\beta_n}, 1\right\}$ and*

$$L = \left\{ l; 1 < l < n, \text{ and } P_S(\vec{\alpha} \to \vec{\beta}) = \frac{E_l(\vec{\alpha})}{E_l(\vec{\beta})} \right\}. \tag{16}$$

The vector $\vec{\alpha}$ serves as a probabilistic self-catalyst for the transformation from $|\alpha\rangle$ to $|\beta\rangle$ if, and only if, for all $r_1, r_2, ..., r_n \in L \cup \{n+1\}$ satisfying $r_1 \geq r_2 \geq ... \geq r_n \neq n+1$, there exist i and j, with $1 \leq j < i \leq n$, such that

$$\frac{\alpha_i}{\alpha_j} < \frac{\beta_{r_j}}{\beta_{r_i-1}} \tag{17}$$

or

$$\frac{\alpha_i}{\alpha_j} > \frac{\beta_{r_j-1}}{\beta_{r_i}}. \tag{18}$$

$\sharp$ Copies (N)	$P_S(\lvert\alpha\rangle \otimes \lvert\alpha\rangle^{\otimes N} \to \lvert\beta\rangle \otimes \lvert\alpha\rangle^{\otimes N})$
0	$\simeq 0.800$
1	$\simeq 0.889$
2	$\simeq 0.907$
3	$\simeq 0.926$
4	$\simeq 0.932$
5	$\simeq 0.940$
6	$\simeq 0.944$
7	$\simeq 0.947$
8	$\simeq 0.950$
9	$\simeq 0.952$
10	$\simeq 0.955$

TABLE II: Increase of optimal probability for the conversion $|\alpha\rangle \to |\beta\rangle$, where $\vec{\alpha} = (0.6, 0.2, 0.2)$ and $\vec{\beta} = (0.5, 0.4, 0.1)$, with the number of copies N of $|\alpha\rangle$ used as a catalyst.

$\sharp$ Copies (N)	$P_S(\lvert\alpha'\rangle \otimes \lvert\alpha'\rangle^{\otimes N} \to \lvert\beta'\rangle \otimes \lvert\alpha'\rangle^{\otimes N})$
0	$\simeq 0.8965$
1	$\simeq 0.9038$
2	$\simeq 0.9072$
3	$\simeq 0.9092$
4	$\simeq 0.9105$
5	$\simeq 0.9109$
6	$\simeq 0.9110$

TABLE III: Increase of optimal probability for the conversion $|\alpha'\rangle \to |\beta'\rangle$, where $\vec{\alpha'} = (0.40, 0.34, 0.15, 0.11)$ and $\vec{\beta} = (0.50, 0.21, 0.17, 0.12)$, with the number of copies N of $|\alpha'\rangle$ used as a catalyst.

Whenever one of the inequalities (17) *or* (18) *has an index* $n+1$*, it is considered to be violated, so the other one must necessarily be satisfied.*

Remark 3. *Again, we observe that typically* L *has only one element* l*, since the minimum which determines the probability of conversion is non-degenerate with probability* 1.

$\sharp$ Copies (N)	$P_S(\lvert\alpha\rangle \otimes \lvert\alpha\rangle^{\otimes N} \to \lvert\beta\rangle \otimes \lvert\alpha\rangle^{\otimes N})$
0	$\simeq 0.600$
1	$\simeq 0.818$
2	$\simeq 0.911$
3	$\simeq 0.957$
4	$\simeq 0.981$
5	$\simeq 0.994$
6	$= 1$

TABLE IV: Increase of optimal probability for the conversion $|\alpha\rangle \to |\beta\rangle$, where $\vec{\alpha} = (0.928, 0.060, 0.006, 0.006)$ and $\vec{\beta} = (0.950, 0.030, 0.0195, 0.0005)$, with the number of copies N of $|\alpha\rangle$ used as a catalyst.

A. Self-Catalysis under SLOCC for random Schmidt vectors

The criterion above, together with Proposition 2 and the behaviour of L, have interesting consequences for the probability of occurrence of self-catalysis.

From Remark 2, we know that the event $[P_S(|\alpha\rangle \to |\beta\rangle) < \min\{\frac{\alpha_n}{\beta n}, 1\}]$ has probability $\gtrsim 1/2$. Conditioning on this event we can then analyze the validity of Ineqs. (17) and (18). With probability 1 we must have $L = \{l\}$, for some $1 < l < n$. Therefore, we can choose the indexes r_i in only two ways: either $r_1 = r_2 = ... = r_n = l$ or $r_1 = r_2 = ... = r_j = n + 1$ and $r_{j+1} = ... = r_n = l$, for some j. For the second case, one can always satisfy one of the inequalities using the index $n + 1$. For the first case, the r.h.s of the inequalities always have index l and we can lower bound the probability for at least one of Inequalities (18) to be valid:

$$\mathbb{P}[\max_{1\leq j<i\leq n}\{\frac{\alpha_i}{\alpha_j}\} > \frac{\beta_{l-1}}{\beta_l}] \geq \mathbb{P}[\max_{1<j\leq n}\{\frac{\alpha_{j-1}}{\alpha_j}\} > \max_{1<j\leq n}\{\frac{\beta_{j-1}}{\beta_j}\}] = 1/2, \tag{19}$$

using that $\max_{1\leq j<i\leq n}\{\frac{\alpha_i}{\alpha_j}\} = \max_{1<j\leq n}\{\frac{\alpha_{j-1}}{\alpha_j}\}$, since $\vec{\alpha}$ is ordered, and the fact that the two random variables on the second term are independent and identically distributed. Putting these together we get that the probability for having SLOCC self-catalysis is $\gtrsim \frac{1}{4}$.

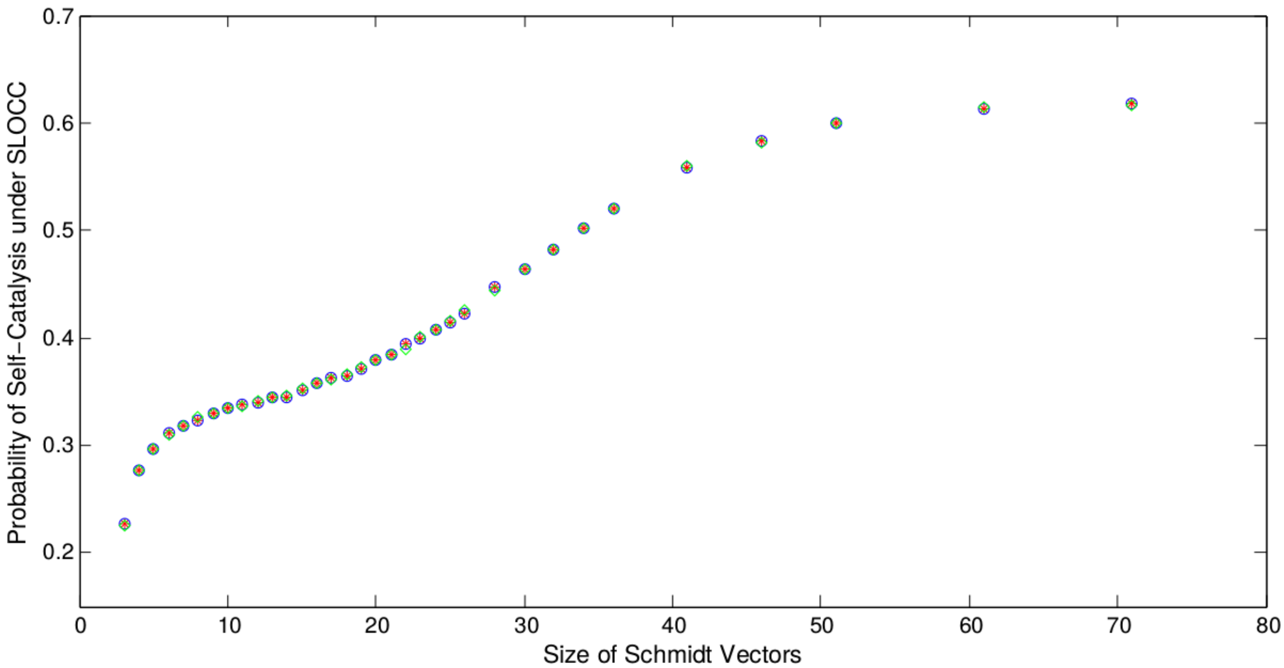

FIG. 5: Probability of finding a pair of states exhibiting self-catalysis under SLOCC, $\mathbb{P}[P_S(|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle) > \mathbb{P}(|\alpha\rangle \to |\beta\rangle)]$, as function of the dimension of each system. Each symbol represents an average over a new set of randomly (Haar) chosen pairs. There are three symbols for each explored dimension.

We have also numerically investigated the typicality of probabilistic self-catalysis by 1) sorting a pair of incomparable Schmidt vectors of same fixed dimension; 2) counting how many of them show the effect; and 3) computing the average gain in probability. To be more specific, we consider as a success case the situation where the pairs are such that $p_1 = P_S(|\alpha\rangle \to |\beta\rangle) < P_S(|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle) = p_2$ and we compute the average value of $p_2 - p_1$. In order to avoid counting cases where $p_1 < p_2$ due to numerical fluctuations, we only consider as valid those vectors where $p_2 > \left(1 + 10^{-5}\right) p_1$. The results are shown in Figures 5 and 6.

First note that they are consisted with the lower bound of 1/4 estimated before. Comparing with the deterministic case, probabilistic self-catalysis is much more frequent, as expected. Even more, here we *do not* have the same qualitative behaviour: the probability of having a pair of incomparable states exhibiting self-catalysis increases monotonically with the size of the Schmidt vectors and seems to be converging to a value around 0.6. Meanwhile, this is not the behaviour of the average probability gain, shown in Figure 6. The average gain in probability is relatively small for all system sizes, and decreases even more for larger sizes. But note that we consider only one copy of $|\alpha\rangle$ attached, and while for many copies the average gain must be larger. It is important to recall Fig. 4, however, which tell us that a

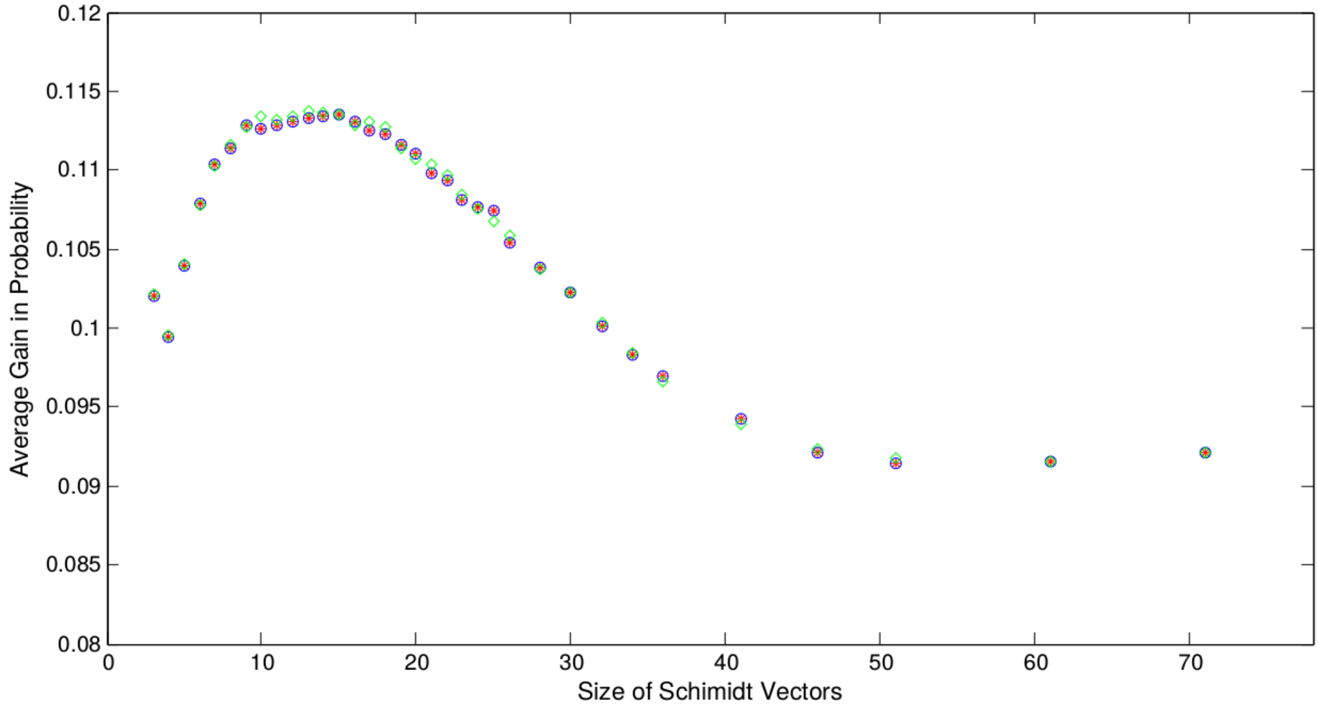

FIG. 6: Average gain of probability, $\mathbb{E}[P_S(|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle) - P_S(|\alpha\rangle \to |\beta\rangle)]$, as a function of the size of Schmidt vectors (sampled following Haar measure), considered only those pairs where self-catalysis occurs, *i.e.*, $P_S(|\alpha\rangle \otimes |\alpha\rangle \to |\beta\rangle \otimes |\alpha\rangle) > P_S(|\alpha\rangle \to |\beta\rangle)$.

pair $\left(\vec{\alpha}, \vec{\beta}\right)$ has, on average, a probability of direct conversion close to 0.6, which naturally bounds the catalytic gain to about 0.4.

Finally, Figure 7 represents, for randomly chosen pairs of incomparable Schmidt vectors with size $n = 45$, the self-catalytic probability gain *versus* direct conversion rates. The diagonal straight line just represents saturation of probability. Some concentration close to the horizontal axis is natural, representing the cases where self-catalysis does not happen. However it is not clear why there is the bold concentrated cloud in red, where the majority of the pairs fit. It empirically means that the most typical situation for a pair of Schmidt vectors of the same size is to have a large probability of conversion and to have a considerable (but not maximal) self-catalytical gain.

VI. CONCLUSIONS

In this paper we have shown the possibility of self-catalytic entanglement conversion for both LOCC and SLOCC scenarios by providing explicit examples of them. We have explored numerically and by arguments of typicality how frequent they are by showing that it is much more common in SLOCC case than the deterministic one, but despite the fact that self-catalysis under SLOCC become more common as the systems sizes increase, the direct self-catalysis decreases with the dimension. Moreover, we also investigated how the phenomenon may depend on the number of copies used as catalyst, finding examples of different behaviours, running from cases where there is no gain in considering multiple copies, to cases where the conversion becomes deterministic for a finite number of copies. Since we rest on numerical techniques, we could not guarantee the existence of a transition where the probability of conversion asymptotically goes to 1. Finally, we computed the average gain in probability in the SLOCC case, showing that this gain, as in the LOCC case, has a global maximum, and also decreases with larger systems sizes.

In answering the question about existence of self-catalysis, we obtained many results, not only on deterministic and probabilistic self-catalysis, but also on ordinary catalysis. About the typicality of self-catalysis, our data support two conjectures: under LOCC, the probability of finding a self-catalytic reaction increases monotonically attaining a global maximum for a dimension about 20. We believe that the origin of this overall non-monotonic behaviour is due the competition between the *sizes* of each Schmidt vectors and the measure *concentration* phenomenon. On the other hand, the data suggests that under SLOCC the probability of finding a self-catalytic reaction increases monotonically with the dimension.

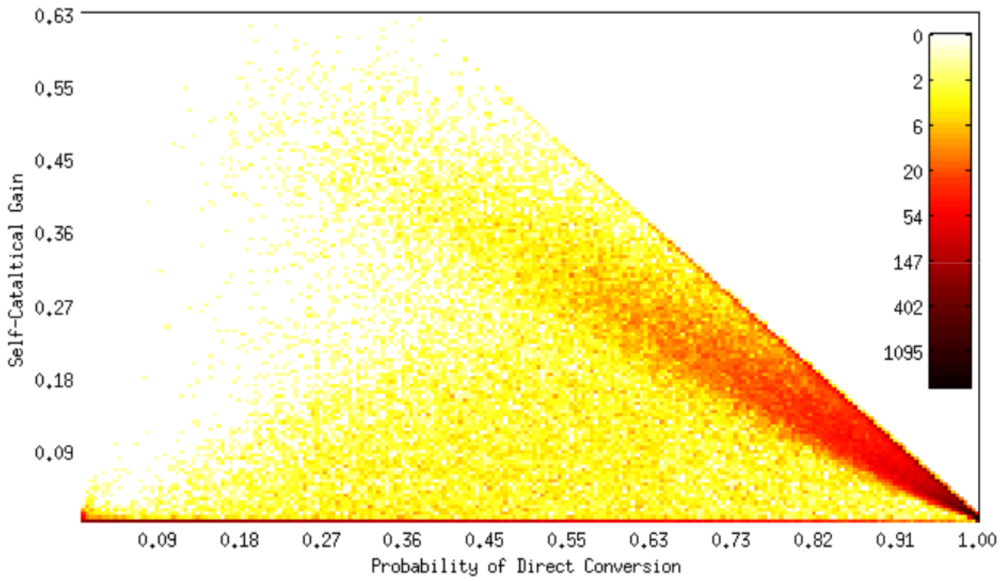

FIG. 7: Self-catalytic probability gain and direct conversion rates for randomly (Haar) chosen pairs of incomparable Schmidt vectors with size $n = 45$. The colour represents the number of pairs per pixel.

In a sense, we estimated numerically the volumes of the sets of pairs of Schmidt vectors where the phenomena take place, but it was not possible to characterize completely the asymptotic behavior with the vectors sizes. In fact, some of our numerical results have a reasonable dependence on the way we sort the random Schmidt vectors. That is something to be explored elsewhere. Perhaps a better understanding of the geometry of the sets involved could help in that analysis.

Acknowledgments

We would like to thank Thaís Matos Acácio, José Amâncio, Luiz Fernando Faria, Eduardo Mascarenhas, Fernando de Melo, Roberto Oliveira, and the anonymous referee for useful comments and stimulating discussions, as well as CNPq, CAPES and FAPEMIG for financial support.

[1] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[2] M. A. Nielsen, *Phys. Rev. Lett.* **83**, 436 (1999).
[3] Eric Chitambar, Debbie Leung, Laura Mančinska, Maris Ozols, Andreas Winter, *Commun. Math. Phys.* **328**(1), pp. 303-326 (2014).
[4] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
[5] B. Liu, J. Li, X. Li, and C. Qiao, *Phys. Rev. Lett.* **108**, 050501 (2012).
[6] A. Miyake, *Int. J. Quant. Info.* **2**, 65-77 (2004).
[7] J. Eisert and M. Wilkens *Phys. Rev. Lett.* **85**, 437 (2000).
[8] M. A. Nielsen, *An introduction to majorization and its applications to quantum mechanics.* Lecture Notes, Departament of Physics, University of Queensland (2002).
[9] D. Jonathan and M. B. Plenio, *Phys. Rev. Lett.* **83**, 3566 (1999).
[10] C. N. Gagatsos, O. Oreshkov, and N. J. Cerf, *Phys. Rev. A* **87**, 042307 (2013).
[11] F. G. Brandão, M. Horodecki, N. H. Ng, J Oppenheim, and S. Wehner, *PNAS* **112**, 3275 (2015).
[12] H. Bragança, et. al., *Phys. Rev. B*, **89** (2014).
[13] X. Sun, R. Duan, and M. Ying, *IEEE Transactions on Information Theory* **51**, p. 1090, (2005).
[14] W. Dür, G. Vidal, and J. I. Cirac, *Phys. Rev. A* **62** (2000).
[15] G. Vidal, *Phys. Rev. Lett.* **83**, 1046 (1999).
[16] Y. Feng, R. Duan and M. Ying, *Phys. Rev. A* **69**, 062310 (2004).
[17] Y. Feng, R. Duan and M. Ying, *IEEE Transactions on Information Theory* **51**, NO. 3, (2005).

[18] M. Ledoux, *The concentration of measure phenomenon* (American Mathematical Society, USA, 2001).
[19] D. N. Page, *Phys. Rev. Lett.* **71**, 1291 (1993).
[20] S. K. Foong and S. Kanno, *Phys. Rev. Lett.* **72**, 1148 (1994).
[21] S. Lloyd and H. Pagels, Ann. Phys. **188**, 186 (1988).
[22] I. Bengtsson and Karol Życzkowski *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, 2008).